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2002 J. Phys. A: Math. Gen. 35 2947

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Separation of variables in quasi-potential systems of bi-cofactor form

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Received 21 November 2001

Published 15 March 2002

Online at stacks.iop.org/JPhysA/35/2947

Abstract

We perform variable separation in the quasi-potential systems of equations of the form $\dot{q} = -A^{-1}\nabla k = -\tilde{A}^{-1}\nabla\tilde{k}$, where A and \tilde{A} are Killing tensors, by embedding these systems into a bi-Hamiltonian chain and by calculating the corresponding Darboux–Nijenhuis coordinates on the symplectic leaves of one of the Hamiltonian structures of the system. We also present examples of the corresponding separation coordinates in two and three dimensions.

PACS numbers: 02.30.lk, 02.40.–k, 45.20.Jj

Mathematics Subject Classification: 70H06, 70H20, 53D17

1. Introduction

In recent years, a new constructive separability theory, based on a bi-Hamiltonian property of integrable systems, was presented. In the frame of canonical coordinates the theory was developed in a series of papers [1–6] (see also the review paper [7]), while a general case was considered in [8–10].

In this paper we apply the bi-Hamiltonian separability theory to solve Hamilton–Jacobi (HJ) equations for a class of quasi-potential systems, called bi-cofactor systems, systematically studied in recent papers [11–15].

The structure of the paper is as follows. In section 2 we briefly describe the one-Casimir bi-Hamiltonian separability theory in the case of bi-Hamiltonian systems with quadratic in momenta constants of motion and relate this theory to the classical Stäckel theory. In section 3 we present basic facts about quasi-potential systems and their special subclass called bi-cofactor systems. Then, in section 4, an explicit form of transformation to separated coordinates is derived, proving the separability of all nondegenerated bi-cofactor systems.

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We sketch the proofs of more important results of this section independently of the general statements of section 2 in order to make this paper more self-contained, but we also point out the places where the general theory and our calculations meet. Finally, in section 5, we illustrate the obtained formulae by some examples.

2. Stäckel separability of one-Casimir bi-Hamiltonian chains

Let us consider a Poisson manifold \mathcal{M} of $\dim \mathcal{M} = 2n+1$ equipped with a linear Poisson pencil $\Pi_\xi = \Pi_1 - \xi \Pi_0$ of maximal rank, i.e. a pair of Poisson operators (tensors) $\Pi_i : T^*\mathcal{M} \rightarrow T\mathcal{M}$ each of rank $2n$ such that their linear combination $\Pi_1 - \xi \Pi_0$ is itself a Poisson operator for any $\xi \in \mathbb{R}$ (the operators Π_0 and Π_1 are then said to be compatible).

Definition 1. A scalar function $h : \mathcal{M} \rightarrow \mathbb{R}$ is called a Casimir function (or a Casimir) of a Poisson operator Π acting in \mathcal{M} if $\Pi \circ dh = 0$.

Here and in what follows $d : C^\infty(\mathcal{M}) \rightarrow \Gamma(T^*\mathcal{M})$ is the operator of the external derivative (gradient operator) in \mathcal{M} ; $\Gamma(T^*\mathcal{M})$ denotes sections of $T^*\mathcal{M}$ (vector fields on \mathcal{M}) and the symbol \circ denotes the composition of mappings. If, as often happens, the Casimir h_ξ of the pencil Π_ξ is a polynomial in ξ of order n

$$h_\xi = h_n \xi^n + h_{n-1} \xi^{n-1} + \dots + h_0 \quad (1)$$

then by expanding the equation $\Pi_\xi \circ dh_\xi = 0$ in powers of ξ and comparing the coefficients at equal powers we obtain the following *bi-Hamiltonian chain*:

$$\begin{aligned} \Pi_1 \circ dh_0 &= 0 \\ \Pi_1 \circ dh_1 &= \Pi_0 \circ dh_0 \\ &\vdots \\ \Pi_1 \circ dh_n &= \Pi_0 \circ dh_{n-1} \\ 0 &= \Pi_0 \circ dh_n. \end{aligned} \quad (2)$$

From (2) it follows that the functions h_i are in involution with respect to both Poisson structures. If additionally all h_i are functionally independent, then the chain is a Liouville integrable system.

Let us now consider a set of coordinates $\{\lambda_i, \mu_i\}_{i=1}^n$ and a Casimir coordinate $c' = h_n$ on \mathcal{M} so that $\{\lambda_i, \mu_i\}_{i=1}^n$ are canonical with respect to Π_0 , so that

$$\Pi_0 = \left[\begin{array}{cc|c} 0 & I & 0 \\ -I & 0 & 0 \\ \hline * & & 0 \end{array} \right].$$

We can try to linearize the system (2) through a canonical transformation $(\lambda, \mu) \rightarrow (b, a)$ in the form $b_i = \frac{\partial W}{\partial a_i}$, $\mu_i = \frac{\partial W}{\partial \lambda_i}$, where $W(\lambda, a)$ is a generating function that solves the related Hamilton–Jacobi (HJ) equations

$$h_r \left(\lambda, \frac{\partial W}{\partial \lambda} \right) = a_r \quad r = 0, \dots, n. \quad (3)$$

In general, the HJ equations (3) are nonlinear partial differential equations that are very difficult to solve. However, there are rare cases when one can find a solution of (3) in the separable form

$$W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a) \quad (4)$$

that turns the HJ equations into a set of decoupled ordinary differential equations that can be solved by quadratures. Such (λ, μ) coordinates are called *separated coordinates*. In the (a, b) coordinates the flow d/dt_j associated with every Hamiltonian h_j is trivial,

$$\frac{da_i}{dt_j} = 0 \quad \frac{db_i}{dt_j} = \delta_{ij} \quad \frac{dc'}{dt_j} = 0 \quad i, j = 1, \dots, n \tag{5}$$

and the implicit form of the trajectories $\lambda(t_j)$ is given by

$$b_i(\lambda, a) = \frac{\partial W}{\partial a_i} = \delta_{ij}t_j + \text{const} \quad i, j = 1, \dots, n. \tag{6}$$

Theorem 2. *A sufficient condition for (λ, μ) to be separated coordinates for the bi-Hamiltonian chain (2) is*

$$H_i(\lambda, \mu, c') = f_i(\lambda_i, \mu_i) \quad i = 1, \dots, n \tag{7}$$

where

$$H_i(\lambda, \mu, c') \equiv c'\lambda_i^n + h_{n-1}(\lambda, \mu, c')\lambda_i^{n-1} + \dots + h_0(\lambda, \mu, c')$$

and $f_i(\lambda_i, \mu_i)$ is some smooth function of a pair of canonically conjugate coordinates λ_i, μ_i .

We will briefly sketch the proof here. Condition (7) can be presented in a matrix form

$$\begin{pmatrix} \lambda_1^n & \lambda_1^{n-1} & \dots & 1 \\ \lambda_2^n & \lambda_2^{n-1} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_n^n & \lambda_n^{n-1} & \dots & 1 \end{pmatrix} \begin{pmatrix} c \\ h_{n-1} \\ \vdots \\ h_0 \end{pmatrix} = \begin{pmatrix} f_1(\lambda_1, \mu_1) \\ f_2(\lambda_2, \mu_2) \\ \vdots \\ f_n(\lambda_n, \mu_n) \end{pmatrix} \quad \text{or} \quad wh = f \tag{8}$$

which is called a *generalized Stäckel representation*. Multiplying the HJ equations (3) written in the matrix form,

$$h = a \quad \text{with} \quad h = (c, h_{n-1}, \dots, h_0)^T \quad \text{and} \quad a = (c, a_{n-1}, \dots, a_0)^T \tag{9}$$

from the left by w one gets $wh = wa$, or according to condition (8), $f = wa$, i.e.

$$f_i \left(\lambda_i, \frac{\partial W}{\partial \lambda_i} \right) = c\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \dots + a_0 \quad i = 1, \dots, n \tag{10}$$

which implies the existence of the separated solution (4) for the system of HJ equations (3). In consequence, the system (3) splits into a decoupled set of ODEs,

$$f_i \left(\lambda_i, \frac{dW_i}{d\lambda_i} \right) = c\lambda_i^n + a_{n-1}\lambda_i^{n-1} + \dots + a_0 \quad i = 1, \dots, n. \tag{11}$$

which concludes the proof.

Let us now restrict to a special case, when $f_i(\lambda_i, \mu_i)$ is quadratic in momenta μ_i

$$f_i(\lambda_i, \mu_i) = f_i(\lambda_i)\mu_i^2 + g_i(\lambda_i) \tag{12}$$

(for motivation of this choice see the end of this section), i.e. to the case when $\mathcal{M} = T^*Q \times \mathbb{R}$, where Q is some Riemannian (pseudo-Riemannian) manifold and T^*Q its cotangent bundle. Then condition (8) can be put in the form

$$\begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} h_{n-1} \\ h_{n-2} \\ \vdots \\ h_0 \end{pmatrix} = \begin{pmatrix} f_1(\lambda_1)\mu_1^2 + g_1(\lambda_1) - \lambda_1^n c \\ f_2(\lambda_2)\mu_2^2 + g_2(\lambda_2) - \lambda_2^n c \\ \vdots \\ f_n(\lambda_n)\mu_n^2 + g_n(\lambda_n) - \lambda_n^n c \end{pmatrix}. \tag{13}$$

This is a system of linear equations with respect to h_i . It has the solution [2]

$$h_i = \sum_{k=1}^n G_i^{kk}(\lambda) \mu_k^2 + V_i(\lambda) + c\rho_i(\lambda) \quad i = 0, \dots, n - 1 \tag{14}$$

where

$$G_i^{kk}(\lambda) = -\frac{\partial \rho_i}{\partial \lambda_k} \frac{f_k(\lambda_k)}{\Delta_k} \quad V_i(\lambda) = -\sum_{k=1}^n \frac{\partial \rho_i}{\partial \lambda_k} \frac{g_k(\lambda_k)}{\Delta_k}$$

$\Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j)$, and $\rho_i(\lambda)$ are the polynomials of order n in λ , defined by the relation

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \sum_{i=0}^n \rho_i \lambda^i \tag{15}$$

(i.e. Viète polynomials). It can be shown that in the language of Riemannian geometry G_i are second-order contravariant tensors on Q ; the tensor G_{n-1} represents a contravariant metric tensor, i.e. the inverse of a standard covariant metric g on Q : $G_{n-1} = g^{-1}$, while the remaining $n - 1$ ones are Killing tensors. From (13) it follows that the constants of motion for the geodesic case have the form

$$\sum_{k=1}^n G_i^{kk}(\lambda) \mu_k^2 = \sum_{k=1}^n (\varphi^{-1})_i^k \mu_k^2 \tag{16}$$

where $\varphi_i^k = \lambda_i^{n-k} / f_i(\lambda_i)$ is a classical Stäckel matrix. It is worth mentioning here that the theory of separation of variables on Riemannian manifolds has been developed by Eisenhart [16] who used the concept of Killing tensors to analyse integrals of geodesic motion. The intrinsic formulation of Eisenhart’s theory has been presented in [17] where the concept of the Killing web was introduced, while the application of the theory to bi-Hamiltonian systems has been presented in [18].

From (2) it can be shown that in (λ, μ) coordinates the operators Π_0 and Π_1 attain the form

$$\Pi_0 = \left[\begin{array}{cc|c} 0 & I & 0 \\ -I & 0 & 0 \\ \hline * & & 0 \end{array} \right] \quad \Pi_1 = \left[\begin{array}{cc|c} 0 & \Lambda & \partial h_{n-1} / \partial \mu \\ -\Lambda & 0 & -\partial h_{n-1} / \partial \lambda \\ \hline * & & 0 \end{array} \right] \tag{17}$$

with the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ and where the symbol $*$ denotes the elements that make the matrices skew-symmetric. In fact there exists a whole family of separated coordinates (λ', μ', c') which preserve the forms (14) and (17) of h_i and Π_i and which are related to the set (λ, μ, c') by a canonical (gauge) transformation

$$\lambda'_i = \pm \lambda_i \quad \mu'_i = \pm \mu_i + \vartheta_i(\lambda_i) \quad i = 1, \dots, n \tag{18}$$

where ϑ is an arbitrary smooth function. In the separated coordinates the Poisson pencil Π_ξ and the chain (2) can be projected onto symplectic leaf $S_{c'}$ of Π_0 ($\dim S = 2n$) producing a nondegenerated Poisson pencil on $S_{c'}$ of the form $\theta_\xi = \theta_1 - \xi \theta_0$, where

$$\theta_0 = \left[\begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right] \quad \theta_1 = \left[\begin{array}{cc} 0 & \Lambda \\ -\Lambda & 0 \end{array} \right]. \tag{19}$$

Hence, $S_{c'}$ is a Poisson–Nijenhuis manifold where the related Nijenhuis tensor \mathcal{N} and its adjoint \mathcal{N}^* ,

$$\mathcal{N} = \theta_1 \circ \theta_0^{-1} = \left[\begin{array}{cc} \Lambda & 0 \\ 0 & \Lambda \end{array} \right] \quad \mathcal{N}^* = \theta_0^{-1} \circ \theta_1 = \left[\begin{array}{cc} \Lambda & 0 \\ 0 & \Lambda \end{array} \right] \tag{20}$$

are diagonal (here we use the name of Nijenhuis tensor for a second-order tensor with a vanishing Nijenhuis torsion). This motivates the following definition [19]:

Definition 3. *The coordinates (λ, μ, c') in which Π_0 and Π_1 have the form (17) (or, equivalently, in which the tensor $\mathcal{N} = \theta_1 \circ \theta_0^{-1}$ has the diagonal form (20)), are called the Darboux–Nijenhuis (DN) coordinates.*

The tensors \mathcal{N} and \mathcal{N}^* are not equal, since \mathcal{N} acts on the space of vector fields while \mathcal{N}^* acts on the space of one-forms on \mathcal{M} . Note that $\rho_i(\lambda)$ are coefficients of minimal polynomial of the Nijenhuis tensor

$$(\det(\lambda I - \mathcal{N}))^{1/2} = \det(\lambda I - \Lambda) = \prod_{i=1}^n (\lambda - \lambda_i) = \sum_{i=0}^n \rho_i \lambda^i \quad \rho_n = 1. \tag{21}$$

Let us now perform an arbitrary, not necessary canonical, c' -independent, nondegenerate coordinate transformation $(\lambda, \mu) \rightarrow (q, p)$, preserving a quadratic dependence on momenta in all $h_k, k = 0, \dots, n - 1$. It can be shown that after such transformation, i.e. in the variables (q, p, c') , the operators Π_0 and Π_1 will have the following form:

$$\Pi_0 = \left[\begin{array}{cc|c} 0 & -\tilde{G}(q) & 0 \\ \tilde{G}^T(q) & \tilde{F}(q, p) & 0 \\ \hline & * & 0 \end{array} \right] \quad \Pi_1 = \left[\begin{array}{cc|c} 0 & G(q) & \\ -G^T(q) & F(q, p) & \\ \hline & * & \end{array} \right] \Pi_0 \circ dh_{n-1} \tag{22}$$

and the minimal polynomial of the Nijenhuis tensor becomes

$$(\det(\xi I - \mathcal{N}))^{1/2} = \frac{\det(\xi \tilde{G} + G)}{\det \tilde{G}}. \tag{23}$$

Obviously, in a real situation we start from a given bi-Hamiltonian chain (2) in arbitrary coordinates (q, p, c') , derived by some method, and find the appropriate transformation to DN coordinates (λ, μ, c') . In a typical situation one starts with the Hamiltonians h_i that written in the coordinates (q, p, c') are quadratic in momenta p . If the transformation to (λ, μ, c') variables is such that λ depends on q only, then the functions f_i must have the form (12). This is precisely the situation one encounters in the case of bi-cofactor systems. In the next sections we will apply the ideas of this section in order to separate this class of systems.

3. Basic facts about bi-cofactor systems

Let us consider the systems of differential equations (in the flat space \mathbb{R}^n) of Newton form

$$\ddot{q} = M(q) \tag{24}$$

where $q = (q_1, \dots, q_n)^T$, $M(q) = (M_1(q), \dots, M_n(q))^T$ where T denotes the transpose of a matrix, $q_i = q_i(t)$, with $t \in \mathbb{R}$ being an independent variable and where overdots denote differentiation with respect to t , so that $\ddot{q}_i = d^2q_i/dt^2$ etc. A simple lemma below focuses our attention on those equations of type (24) that posses a quadratic in \dot{q} integral of motion.

Lemma 4. *Let*

$$E(q, \dot{q}) = \dot{q}^T A(q) \dot{q} + k(q) = \sum_{i,j=1}^n \dot{q}_i A_{ij}(q) \dot{q}_j + k(q)$$

where $A = A^T$ is a nondegenerated, $n \times n$ symmetric matrix with q -dependent entries. Then E is an integral of motion for the system (24) (that is $\dot{E} = 0$) if and only if the following two equations hold:

$$\partial_i A_{jk} + \partial_j A_{ki} + \partial_k A_{ij} = 0 \quad \text{for all } i, j, k = 1, \dots, n \tag{25}$$

$$2AM + \nabla k = 0 \quad (26)$$

Here and in what follows $\partial_i = \partial/\partial q_i$ and $\nabla k(q) = (\partial_1 k, \dots, \partial_n k)$. This lemma can easily be proved by calculating the derivative $\dot{E} \equiv dE/dt$.

Equation (25) implies that the matrix A is a Killing tensor. We will restrict ourselves to a class of solutions of (25) that have the form

$$A = \text{cof}(G) \quad (27)$$

with

$$G = \alpha q q^T + \beta q^T + q \beta^T + \gamma \quad (28)$$

where cof means the cofactor matrix (so that $\text{cof}(G)G = \det(G)$ or, in the case when G is invertible, $\text{cof}(G) = \det(G)G^{-1}$), α is a real constant, $\beta = (\beta_1, \dots, \beta_n)^T$ is a column vector of constants and where γ is a symmetric $n \times n$ constant matrix. It is easy to show that for $n = 2$ it is the general solution of (25). For higher n the general solution of (25) depends on $n(n+1)^2(n+2)/12$ parameters (its basis can be found e.g. in [20]) while (27) has only $1+n+n(n+1)/2$ parameters and is therefore far from being a general solution. It turns out, however, that this particular solution has interesting properties that make it worth studying. It originates in a natural way when one considers a broad class of Poisson pencils of type (34) (see [12]). It leads to the notion of bi-cofactor systems that admit many differential-algebraic properties, among them interesting recursion formulae that allow the generation of nontrivial bi-cofactor systems from a trivial (geodesic) flow (see below). It has been recently generalized to the case of Riemannian manifolds [21]. In this more general setting it can be demonstrated that the matrix G given by (28) is a conformal Killing tensor, which leads to the orthogonal separability of the geodesic Hamilton–Jacobi equation [18].

Equation (26) implies that whenever $\det(A) \neq 0$ the force M can be written in the *quasi-potential* form $M = -\frac{1}{2}A^{-1}\nabla k$, which generalizes the usual potential case and which reduces to the potential case when $A = \frac{1}{2}I$, where I stands for the identity matrix. Clearly, in the case when our system (24) has the second—functionally independent of E —integral of motion of the form

$$\tilde{E}(q, \dot{q}) = \dot{q}^T \tilde{A}(q)\dot{q} + \tilde{k}(q)$$

with an invertible matrix \tilde{A} , then it can be written in a quasi-potential form in two distinct ways. It motivates the following definition:

Definition 5. A system of equations

$$\ddot{q} = M(q) = -\frac{1}{2}A^{-1}(q)\nabla k(q) = -\frac{1}{2}\tilde{A}^{-1}(q)\nabla \tilde{k}(q) \quad (29)$$

where A and \tilde{A} are two linearly independent matrices of the cofactor form (27)–(28):

$$\begin{aligned} A &= \text{cof}(G) & G &= \alpha q q^T + \beta q^T + q \beta^T + \gamma \\ \tilde{A} &= \text{cof}(\tilde{G}) & \tilde{G} &= \tilde{\alpha} q q^T + \tilde{\beta} q^T + q \tilde{\beta}^T + \tilde{\gamma} \end{aligned}$$

and where $k = k(q)$ and $\tilde{k} = \tilde{k}(q)$ are two scalar functions, is called a bi-cofactor system.

The bi-cofactor systems were first studied in [11, 12] and in [13], where they were called cofactor pair systems. In this paper we will deal exactly with this type of system: we will show how to perform separation of variables for quasi-potential systems of the bi-cofactor form.

An important property of such systems is that they actually admit n constants of motion that are quadratic in \dot{q} .

Theorem 6 (Hans Lundmark [13]). *If the Newton system (24) has a bi-cofactor form (29), then it has n integrals of motion of the form*

$$E_i(q, \dot{q}) = \dot{q}^T A_i(q) \dot{q} + k_i(q) \quad i = 0, \dots, n - 1 \tag{30}$$

where the matrices A_i are defined as coefficients in the polynomial expansion of $\text{cof}(G + \xi \tilde{G})$ with respect to the real parameter ξ :

$$\text{cof}(G + \xi \tilde{G}) = \sum_{i=0}^{n-1} A_i \xi^i$$

with $A_0 = \text{cof}(G)$, $A_{n-1} = \text{cof}(\tilde{G})$ and where $k_0 = k$ and $k_{n-1} = \tilde{k}$. Consequently, in the case when all matrices A_i are invertible, such a system can be written in a quasi-potential form in n distinct ways:

$$\ddot{q} = M(q) = -\frac{1}{2} A_i^{-1} \nabla k_i \quad i = 0, \dots, n - 1. \tag{31}$$

Remark 7. In the notation as above we have of course $E_0 = E$ and $E_{n-1} = \tilde{E}$.

Of course, for a given pair of matrices G and \tilde{G} not every function k will have a counterpart \tilde{k} that will satisfy equation (29). However, there exists a recursion formula that from a given bi-cofactor system produces a new bi-cofactor system.

Proposition 8. *Let $\ddot{q} = M(q)$ be a bi-cofactor system of the form (29) with the integrals given by (30). Let also $k_\xi = \sum_{i=0}^{n-1} k_i \xi^i$ (with $k_{n-1} = \tilde{k}$). Then the functions l_i , $i = 0, \dots, n - 1$, defined as $l_\xi = \sum_{i=0}^{n-1} l_i \xi^i$ through*

$$l_\xi = \frac{\det(G + \xi \tilde{G})}{\det(\tilde{G})} \tilde{k} - \xi k_\xi \tag{32}$$

satisfy the relation $A_i^{-1} \nabla l_i = A_j^{-1} \nabla l_j$ for all $i, j = 0, \dots, n - 1$ and are in consequence right-hand sides of a new bi-cofactor system of the form (31) (with l_i instead of k_i).

The proof of this statement can be found in [13]. This formula makes it possible to produce infinite sequences of bi-cofactor systems starting for example from a simple geodesic equation $\ddot{q} = 0$ which is obviously of a bi-cofactor form with, for example, $k_0 = 0$ and $k_{n-1} = 1$. Also, it can easily be inverted in order to express old quasi-potentials k_i through the new quasi-potentials l_i :

$$k_\xi = \frac{1}{\xi} \left(\frac{\det(G + \xi \tilde{G})}{\det(G)} l - l_\xi \right). \tag{33}$$

In consequence, by using (33) it is possible to produce ‘lower’ systems from the ‘higher’ ones, and if we start from the geodesic equation $\ddot{q} = 0$, we obtain in general the family of ‘negative’ systems different from the sequence of systems obtained from $\ddot{q} = 0$ by the use of (32).

It has been proved [12, 13] that the system (29) can be embedded in a bi-Hamiltonian system. In order to make this statement more precise, let us consider a following skew-symmetric operator pencil associated with the system (29):

$$\begin{aligned} \Pi_\xi = \Pi_1 - \xi \Pi_0 \equiv & \left[\begin{array}{cc|c} 0 & G(q) & p \\ -G(q) & F(q, p) & M(q) + 2cN(q) \\ \hline & * & 0 \end{array} \right] \\ & - \xi \left[\begin{array}{cc|c} 0 & -\tilde{G}(q) & 0 \\ \tilde{G}(q) & -\tilde{F}(q, p) & -2c\tilde{N}(q) \\ \hline & * & 0 \end{array} \right] \end{aligned} \tag{34}$$

where both Π_1 and Π_0 are $(2n+1) \times (2n+1)$ matrices/operators acting in the $(2n+1)$ -dimensional space $\mathcal{M} = \mathbb{R}^{2n+1}$ with Cartesian coordinates labelled with (q, p, c) where $q = (q_1, \dots, q_n)^T$, $p = (p_1, \dots, p_n)^T$, $c \in \mathbb{R}$. The $n \times n$ symmetric matrices G and \tilde{G} are exactly the matrices that define our system (29). The $n \times 1$ matrices N and \tilde{N} are given by

$$N = \alpha q + \beta \quad \tilde{N} = \tilde{\alpha} q + \tilde{\beta}.$$

The $n \times n$ matrices F and \tilde{F} are defined by

$$F = Np^T - pN^T \quad \tilde{F} = \tilde{N}p^T - p\tilde{N}^T.$$

As usual, the asterisk $*$ denotes the elements that make our matrices skew-symmetric. This operator pencil is a Poisson pencil precisely due to the fact that the term $M(q)$ can be represented as $M(q) = -\frac{1}{2}A_i^{-1}\nabla k_i$ for all $i = 0, \dots, n-1$. One should also note that this pencil is linear in the variable c and that it has a maximal rank $2n$. The Casimir function h_ξ (see definition 1; obviously, in our variables $d = (\partial/\partial q, \partial/\partial p, \partial/\partial c)^T$) of our Poisson operator Π_ξ is a polynomial of grad n in ξ . In fact, the Casimir h_ξ of Π_ξ has the form

$$h_\xi(q, p, c) = p^T \operatorname{cof}(G + \xi \tilde{G})p + k_\xi(q) - 2c \det(G + \xi \tilde{G}) \quad (35)$$

where $k_\xi(q) = \sum_{i=0}^{n-1} k_i(q)\xi^i$ are as in proposition 8 above. Thus, $h_\xi = \sum_{i=0}^n h_i \xi^i$ with the functions $h_i(q, p, c)$ given explicitly by

$$h_i(q, p, c) = E_i(q, p) - 2cD_i \quad i = 0, \dots, n-1 \quad h_n(q, c) = -2cD_n \quad (36)$$

with $D_i = D_i(q)$ defined as

$$\sum_{i=0}^n D_i(q)\xi^i = \det(G + \xi \tilde{G})$$

so that $D_0 = \det(G)$ and $D_n = \det(\tilde{G})$. The functions E_i in (36) are just the constants of motion (30) of our system (29). By expanding the equation $\Pi_\xi \circ dh_\xi = 0$ in powers of ξ and comparing the coefficients at equal powers we obtain the *bi-Hamiltonian chain* of the form (2) which by the theorem of Magri [22] is completely integrable in the sense of Liouville. This means that the evolutionary equations

$$\frac{d}{dt} \begin{bmatrix} q \\ p \\ c \end{bmatrix} = \Pi_1 \circ dh_i = \Pi_0 \circ dh_{i-1} \quad i = 1, \dots, n \quad (37)$$

associated with (2) are Liouville-integrable.

Let us now investigate the relation of our chain (2) (or of our Poisson pencil (34)) to the system (29). One can show that the last equation in (37) has at the hyperplane $c = 0$ the following form:

$$\frac{d}{dt} \begin{bmatrix} q \\ p \\ c \end{bmatrix} = -2 \det(\tilde{G}) \begin{bmatrix} p \\ M \\ 0 \end{bmatrix} \quad (38)$$

so that the hyperplane $c = 0$ is invariant with respect to this equation. On the other hand, if we set $\dot{q} = p$ in (29), we obtain its equivalent form

$$\frac{d}{dt} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} p \\ M \end{bmatrix} \quad (39)$$

which differ from (38) only by the coefficient $-2 \det(\tilde{G})$. It means that both systems have the same trajectories in the (q, p) -space, although traversed at different speed.

Lemma 9 (rescaling). *Let $x(t) = (x_1(t), \dots, x_n(t))$ be a solution of a first-order differential equation $\dot{x} = X(x)$ in \mathbb{R}^n with $x(0) = x_0 \in \mathbb{R}^n$ and let $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}$ be a scalar function. Then the function $y(t) = x(\tau^{-1}(t))$ with $\tau(t) = \int \frac{1}{\alpha(x(t))} dt$ (chosen so that $\tau(0) = 0$) solves the differential equation $\dot{y} = \alpha(y)X(y)$ with the initial condition $y(0) = x_0$.*

The proof of this lemma is elementary. This lemma implies that knowing a particular solution of (38) we can write down the corresponding solution of (39), just by identifying α with $-1/(2 \det(\tilde{G}))$. In the next section we will show how one can solve all the equations (37) (and thus also the bi-cofactor system (39)) by a procedure that separates variables in Hamilton–Jacobi equations that correspond to all Hamiltonians h_i of the chain (2).

More information about bi-cofactor systems as well as more detailed explanations of the facts mentioned above can be found in [12, 13, 15]. In [21] and [23] one can find a generalization of many of the above-mentioned statements to the case of Riemannian manifolds.

4. Separation of variables

In the preceding section we explained how our bi-cofactor system (29) can be embedded in a bi-Hamiltonian chain (2). The main goal of this paper is to present the procedure that leads to separation of variables in the Hamilton–Jacobi equations corresponding to the Hamiltonians h_i . According to the rescaling lemma above, such a procedure will yield a solution of our bi-cofactor system as well. This procedure is based on the results obtained in [8, 9, 24] and other papers.

Let us begin by adjusting our coordinate system so that one of the new coordinates corresponds to the foliation of \mathcal{M} into symplectic leaves of Π_0 . We will obtain it by rescaling c . Let us thus introduce the following curvilinear coordinate system in \mathcal{M} :

$$q'_i = q_i \quad p'_i = p_i \quad i = 1, \dots, n \quad c' = h_n(q, c) = -2c \det(\tilde{G})$$

(observe that the hypersurfaces $c = 0$ and $c' = 0$ coincide). One should note that this transformation does not depend on a particular choice of k and \tilde{k} (since the operator Π_0 does not depend on k and \tilde{k}) but only on the choice of \tilde{G} . In what follows we will write q_i and p_i instead of q'_i and p'_i . In the q, p, c' -variables the chain (2) has the same form; however, the explicit form of operators Π_0 and Π_1 changes to

$$\Pi_0 = \left[\begin{array}{cc|c} 0 & -\tilde{G} & 0 \\ \tilde{G} & -\tilde{F} & 0 \\ \hline & * & 0 \end{array} \right] \quad \Pi_1 = \left[\begin{array}{cc|c} 0 & G & -2p \det(\tilde{G}) \\ -G & F & 2c'(N - G\tilde{G}^{-1}\tilde{N}) - 2 \det(\tilde{G})M \\ \hline & * & 0 \end{array} \right]$$

(cf (22)), while the Hamiltonians h_i attain the form

$$h_i(q, p, c') = E_i(q, p) + c' \frac{D_i}{D_n} \quad i = 0, \dots, n - 1 \quad h_n = c'.$$

In the new variables the symplectic leaves of Π_0 have the desired form $c' = \text{const}$. Naturally, the question arises, if we could not choose a different pencil Π so that formula (38) would not contain the factor $-2 \det(\tilde{G})$, which would make lemma 9 unnecessary, but we were not able to do so yet.

We are now in a position to present the main theorem of this paper.

Theorem 10. *If the roots $\lambda_i = \lambda_i(q)$ of the equation*

$$\det(G + \lambda \tilde{G}) = 0 \tag{40}$$

are functionally independent, then in the variables $\lambda = (\lambda_1, \dots, \lambda_n)^T$, $\mu = (\mu_1, \dots, \mu_n)^T$, c' given by

$$\lambda_i(q) \text{ as roots of (40)} \tag{41}$$

$$\mu_i(q, p) = -\frac{1}{2} \frac{\Omega^T \operatorname{cof}(G + \lambda_i(q)\tilde{G})p}{\Omega^T \operatorname{cof}(G + \lambda_i(q)\tilde{G})\Omega} \quad i = 1, \dots, n \tag{42}$$

where $\Omega = G\tilde{G}^{-1}\tilde{N} - N$, the operators Π_0 and Π_1 attain the form

$$\Pi_0 = \left[\begin{array}{cc|c} 0 & I & 0 \\ -I & 0 & 0 \\ * & & 0 \end{array} \right] \quad \Pi_1 = \left[\begin{array}{cc|c} 0 & \Lambda & \partial h_{n-1}/\partial \mu \\ -\Lambda & 0 & -\partial h_{n-1}/\partial \lambda \\ * & & 0 \end{array} \right] \tag{43}$$

(cf (22)), with the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, while the Hamiltonians h_i have the form

$$h_i(\lambda, \mu, c') = -\sum_{k=1}^n \frac{\partial \rho_i}{\partial \lambda_k} \frac{f_k(\lambda_k, \mu_k)}{\Delta_k} + c' \rho_i(\lambda) \quad i = 0, \dots, n \quad (n \geq 2) \tag{44}$$

$$h_1(\lambda, \mu, c') = f(\lambda, \mu) + c' \rho_1(\lambda) \quad (n = 1)$$

(cf (14)), with some functions f_k depending only on one pair λ_k, μ_k of the variables, $\Delta_k = \prod_{j \neq k} (\lambda_k - \lambda_j)$, and where $\rho_i(\lambda)$ are the Viète polynomials (15). Moreover, in the variables λ, μ, c' the recursion formula (32) attains the form

$$l_\xi(\lambda) = \det(\xi I - \Lambda)\tilde{k}(\lambda) - \xi k_\xi(\lambda) \tag{45}$$

i.e. formula (32) is invariant with respect to the change of variables $q \rightarrow \lambda$.

Remark 11. It is worth mentioning that the coordinates $\lambda(q)$ defined by (41) are in general not orthogonal, and that the gradients $\nabla \lambda_i(q)$ are eigenvectors of the matrix $G\tilde{G}^{-1}$, i.e. $(G + \lambda_i\tilde{G})\nabla \lambda_i = 0$. Moreover, $\nabla \lambda_i(q)$ are \tilde{G} -orthogonal: $\nabla \lambda_i G \nabla \lambda_j = 0$ for $i \neq j$. In the case when one of the matrices, say \tilde{G} , is equal to the identity matrix (that is when our system (29) becomes potential) the above transformation (41)–(42) reduces to the classical formula for point transformation to separation coordinates for natural Hamiltonian systems.

Remark 12. The terms D_i/D_n in the Hamiltonians h_i attain the form $\rho_i(\lambda)$ in (44) precisely due to the fact that D_i/D_n are coefficients in the polynomial expansion of $\det(G + \lambda\tilde{G})/\det(\tilde{G}) = \sum_{i=0}^n \rho_i \lambda^i$ (cf (23)).

Remark 13. Formulae (40)–(42) provide us with a transformation that is independent of the particular choice of the functions k and \tilde{k} in the bi-cofactor system (29), i.e. this transformation will simultaneously separate all the bi-cofactor systems with the same matrices G and \tilde{G} .

Theorem 10 means that the coordinates (λ, μ, c') are Darboux–Nijenhuis coordinates for our operators Π_0, Π_1 . According to the results of section 2, we have

Corollary 14. The Hamilton–Jacobi equations for the Hamiltonians $h_i(\lambda, \mu, c')$,

$$h_i \left(\lambda, \frac{\partial W}{\partial \lambda}, c' \right) = a_i \quad i = 0, 1, \dots, n$$

where $W(\lambda, a)$ is a generating function for the transformation $(\lambda, \mu) \rightarrow (b, a)$, separate under the ansatz $W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a)$ into a system of ODEs of the form

$$f_k \left(\lambda_k, \frac{dW_k}{d\lambda_k} \right) = c' \lambda_k^n + a_1 \lambda_k^{n-1} + \dots + a_n. \tag{46}$$

Proof. The Hamilton–Jacobi equations for Hamiltonians (44) can be treated as a system of n linear equations for functions f_j . Applying the Cramer rule to this system we arrive at (46). □

Thus, we are able to find W up to quadratures. If we denote the evolution parameter associated with h_j by t_j , then in the new variables (b, a) defined implicitly as usual:

$$b_i = \frac{\partial W(\lambda, a)}{\partial a_i} \quad \mu_i = \frac{\partial W(\lambda, a)}{\partial \lambda_i} \quad i = 1, \dots, n$$

the flow associated with h_j has the trivial form (5) so that the transformation $(\lambda, \mu) \rightarrow (b, a)$ simultaneously trivializes Hamilton equations generated by all the Hamiltonians h_i .

We will now sketch the proof of theorem 10. Consider a symplectic leaf $S_{c'} = \{(q, p, c') : c' = \text{const}\}$ of Π_0 . Let us choose a vector field transversal to the symplectic foliation of Π_0 as $Z = \frac{\partial}{\partial c'}$. It can be shown by direct calculation that

$$L_Z \Pi_0 = 0 \quad L_Z \Pi_1 = X \wedge Z \quad \text{with} \quad X = \Pi_0 \circ d(Z(h_{n-1}))$$

where L_Z is a Lie derivative operator in the direction of the vector field Z . The above relations guarantee that we can perform a projection of both Π_0 and Π_1 onto the symplectic leaf $S_{c'}$ of Π_0 . The obtained $2n$ -dimensional Poisson operators θ_0 and θ_1 have the form

$$\theta_0(q, p) = \begin{bmatrix} 0 & -\tilde{G}(q) \\ \tilde{G}(q) & -\tilde{F}(q, p) \end{bmatrix} \quad \theta_1(q, p) \equiv \begin{bmatrix} 0 & G(q) \\ -G(q) & F(q, p) \end{bmatrix}.$$

The corresponding Nijenhuis tensor $\mathcal{N} = \theta_1 \circ \theta_0^{-1}$ has the minimal polynomial of the form $\det(G + \lambda \tilde{G}) / \det(\tilde{G})$ (cf (23)) and its roots are precisely the roots of (40). On the other hand, the roots of the minimal polynomial of \mathcal{N} define—according to (21)—the first half of the transformation $(q, p) \rightarrow (\lambda, \mu)$ to DN coordinates in which the operators θ_0 and θ_1 have the form (19). Due to the last but one equation in the chain (2) this implies that in the DN coordinates the operators Π_0 and Π_1 must have the form (43).

We will now show that the remaining part of the transformation $(q, p) \rightarrow (\lambda, \mu)$ to the DN coordinates, i.e. the expression for $\mu = \mu(q, p)$, must be of the form (42). We will do it in a few steps.

Lemma 15. *In the DN coordinates the vector field X has a simple form:*

$$X = \sum_{i=1}^n \frac{\partial}{\partial \mu_i}.$$

Proof. It is enough to calculate $X = \Pi_0 \circ d(Z(h_{n-1}))$ in the DN coordinates:

$$d(Z(h_{n-1})) = d\left(\frac{\partial}{\partial c'}(h_{n-1})\right) = d\left(\frac{D_{n-1}}{D_n}\right) = -d(\lambda_1 + \dots + \lambda_n)$$

where the last equality is due to the fact that D_{n-1}/D_n is precisely the same term as the last but one in the polynomial expansion of $\det(G + \lambda \tilde{G}) / \det(\tilde{G})$, which in turn is precisely the Viète polynomial $\rho_{n-1} = -(\lambda_1 + \dots + \lambda_n)$. Thus,

$$\Pi_0 \circ d(Z(h_{n-1})) = -\Pi_0 \circ d(\lambda_1 + \dots + \lambda_n) = \sum_{i=1}^n \frac{\partial}{\partial \mu_i}. \quad \square$$

Lemma 16. *The function*

$$H_i(\lambda, \mu, c') \equiv \sum_{k=0}^n h_k(\lambda, \mu, c') \lambda_i^k$$

(i.e. Casimir (35) written in (λ, μ, c') variables and evaluated at λ_i) depends only on the i th pair λ_i, μ_i of the variables λ, μ i.e.

$$H_i(\lambda, \mu, c') = f_i(\lambda_i, \mu_i)$$

for some function $f_i(\lambda_i, \mu_i)$.

Proof. We have that $\partial H_i / \partial c' = \sum_{k=0}^n \lambda_i^k \partial h_k / \partial c' = \sum_{k=0}^n \lambda_i^k D_k / D_n = \det(G + \lambda_i \tilde{G}) / \det(\tilde{G}) = 0$ due to (40). For $j \neq i$ we observe that $\partial H_i / \partial \lambda_j = \frac{\partial h_0}{\partial \lambda_j} + \lambda_i \frac{\partial h_1}{\partial \lambda_j} + \lambda_i^2 \frac{\partial h_2}{\partial \lambda_j} + \dots + \lambda_i^n \frac{\partial h_n}{\partial \lambda_j}$, $i, j = 1, \dots, n$. On the other hand, due to (43) and (2)

$$-\lambda_j \frac{\partial h_k}{\partial \lambda_j} - \frac{\partial h_{n-1}}{\partial \lambda_j} \frac{\partial h_k}{\partial c'} = -\frac{\partial h_{k-1}}{\partial \lambda_j} \quad k, j = 1, \dots, n$$

so that

$$\frac{\partial h_k}{\partial \lambda_j} = -\frac{\partial h_{n-1}}{\partial \lambda_j} \left(\frac{\rho_k}{\lambda_j} + \frac{\rho_{k-1}}{\lambda_j^2} + \dots + \frac{\rho_0}{\lambda_j^{k+1}} \right)$$

which substituted in the above expression for $\partial H_i / \partial \lambda_j$ yields zero. In a similar way one can prove that $\partial H_i / \partial \mu_j = 0$ for $i \neq j$. \square

By theorem 2 of section 2 (and the pages that follow this theorem) this lemma means that our coordinates λ, μ indeed are separation coordinates for our systems. We will however continue our line of proof of theorem 10.

Corollary 17. *From the above lemma it follows, by Cramer's rule, that h_i must have the form (44).*

Lemma 18 [24]. *Let $X = \Pi_0 d(Z(h_{n-1}))$. Suppose that $X^r(H_i) = 0$ for some $r = 2, 3, \dots$ and that $X^k(H_i) \neq 0$ for $k = 1, \dots, r - 1$. Then*

$$\mu_i = \frac{X^{r-2}(H_i)}{X^{r-1}(H_i)}. \quad (47)$$

In order to prove this lemma it is sufficient to integrate the relation $X^r(H_i) = 0$ twice, using the fact that $X = \sum_{i=1}^n \frac{\partial}{\partial \mu_i}$ and lemma 16 and use gauge invariance of DN coordinates in order to kill integration functions that appear after second integration.

In our case the exponent r that 'kills' H_i is equal to 3, since in our old coordinates (q, p, c') the vector field X has the form $X = \sum_i (\dots) \frac{\partial}{\partial p_i}$ and since h_i are quadratic in p . Explicit calculation of expression (47) for $r = 3$ and in q, p, c' -variables yields exactly (42).

The recursion formula (45) is obtained by inserting I and Λ as $-\tilde{G}$ and G in (32). This concludes the proof of theorem 10.

5. Examples

We will now illustrate the content of the presented theory with examples. It is worth noting that the general theory does not provide us with any tools for calculating the functions f_i in (44) (or in (46)). Instead, we have to calculate these functions each time we perform the variable separation of a given bi-cofactor system. In the case of bi-cofactor systems however it turns out that the functions f_i always have the form (12).

As a first example, let us consider the family of parabolic separable potentials introduced in [25]. They have the form

$$V^{(r)}(q_1, q_2) = \sum_{k=0}^{\lfloor r/2 \rfloor} 2^{r-2k} \binom{r-k}{k} q_1^{2k} q_2^{r-2k}$$

with the other integral of motion given by

$$E^{(r)} = -q_2 \dot{q}_1^2 + q_1 \dot{q}_1 \dot{q}_2 + q_1^2 V^{(r-1)}. \tag{48}$$

We easily find the corresponding matrices G and \tilde{G} :

$$G = \begin{bmatrix} 0 & -q_1/2 \\ -q_1/2 & -q_2 \end{bmatrix} \quad \tilde{G} = \frac{1}{2} I \tag{49}$$

The recursion formula (32), applied to the geodesic equation $\ddot{q} = 0$ with $k_\xi = \xi$ (i.e. with $k_0 = 0, k_1 = 1$) produces an infinite family of pairs of quasi-potentials $k_0^{(r)}, k_1^{(r)}, r = 0, 1, \dots$ (with $k_0^{(0)} = k_0, k_1^{(0)} = k_1$), such that $(\text{cof}(G))^{-1} \nabla k_0^{(r)} = \text{cof}(\tilde{G})^{-1} \nabla k_1^{(r)} \equiv 2 \nabla k_1^{(r)}$. The first few are

$$\begin{aligned} k_0^{(1)} &= -q_1^2 & k_1^{(1)} &= -2q_2 \\ k_0^{(2)} &= 2q_2 q_1^2 & k_1^{(2)} &= q_1^2 + 4q_2^2 \\ k_0^{(3)} &= -q_1^4 - 4q_1^2 q_2^2 & k_1^{(3)} &= -4q_1^2 q_2 - 8q_2^3 \end{aligned} \tag{50}$$

where $k_1^{(r)}$ correspond to the potentials $V^{(r)}$ up to a factor $(-1)^r$ and where according to (48) $k_0^{(r)} = -q_1^2 k_1^{(r-1)}$. The corresponding potential-cofactor systems $\ddot{q} = M^{(r)}(q) = -\frac{1}{2}(\text{cof}(G))^{-1} \nabla k_0^{(r)} = -\nabla k_1^{(r)}$ have the form

$$\begin{aligned} \ddot{q} &= M^{(1)}(q) = (0, 2)^T \\ \ddot{q} &= M^{(2)}(q) = (-2q_1, -8q_2)^T \\ \ddot{q} &= M^{(3)}(q) = (8q_1 q_2, 4q_1^2 + 24q_2^2)^T \end{aligned}$$

so that the third one is already nontrivial. Similarly, by applying formula (33), we can produce from the geodesic equation $\ddot{q} = 0$ with $k_\xi = 1$ (i.e. with $k_0^{(0)} = 1, k_1^{(0)} = 0$) the ‘negative’ quasi-potentials. The first few of them are of the form

$$\begin{aligned} k_0^{(-1)} &= 2q_2/q_1^2 & k_1^{(-1)} &= -1/q_1^2 \\ k_0^{(-2)} &= (4q_2^2 + q_1^2)/q_1^4 & k_1^{(-2)} &= -2q_2/q_1^4 \\ k_0^{(-3)} &= 4q_2(2q_2^2 + q_1^2)/q_1^6 & k_1^{(-3)} &= -(4q_2^2 + q_1^2)/q_1^6 \end{aligned} \tag{51}$$

and correspond to potential-cofactor systems $\ddot{q} = M^{(r)}(q) = -\frac{1}{2}(\text{cof}(G))^{-1} \nabla k_0^{(r)} = -\nabla k_1^{(r)}$ with

$$\begin{aligned} \ddot{q} &= M^{(-1)}(q) = (-2/q_1^3, 0)^T \\ \ddot{q} &= M^{(-2)}(q) = (-8q_2/q_1^5, 2/q_1^4)^T \\ \ddot{q} &= M^{(-3)}(q) = (-4(6q_2^2 + q_1^2)/q_1^7, 8q_2/q_1^6)^T. \end{aligned}$$

In order to check what variables will separate these systems, we have to solve equation (40) with G and \tilde{G} given as in (49). An easy computation yields

$$\lambda_1(q) = q_2 - \sqrt{q_1^2 + q_2^2} \quad \lambda_2(q) = q_2 + \sqrt{q_1^2 + q_2^2}$$

Formula (42) in this case reads

$$\mu_i = \frac{\sqrt{-\lambda_1(q)\lambda_2(q)}}{\lambda_i(q)} p_1 + p_2 \quad i = 1, 2$$

and it can immediately be shown that the above formulae present the classical point transformation to the parabolic coordinates. Thus, not only the potentials $V^{(r)}$ but even the corresponding chain (2) is separable in the parabolic coordinates which is perhaps what we should expect. After some algebraic manipulations, the above formulae can be inverted to

$$\begin{aligned} q_1 &= \sqrt{-\lambda_1 \lambda_2} & q_2 &= \frac{1}{2}(\lambda_1 + \lambda_2) \\ p_1 &= \frac{\sqrt{-\lambda_1 \lambda_2}(\mu_1 - \mu_2)}{\lambda_1 - \lambda_2} & p_2 &= \frac{\mu_1 \lambda_1 - \mu_2 \lambda_2}{\lambda_1 - \lambda_2} \end{aligned}$$

which makes it possible to express the Hamiltonians h_i in the DN coordinates (λ, μ, c') . According to (36) and (30) in the old variables (q, p, c') they have the form

$$h_i = \gamma_i(q, p) + k_i(q) + c' \frac{D_i}{D_n}$$

where $\gamma_i(q, p) = p^T A_i(q) p$ is the geodesic part (geodesic Hamiltonian) of h_i . Terms $\frac{D_i}{D_n} = \rho_i$ have in the DN coordinates the form of Viète polynomials (15) while the form of k_i for a given bi-cofactor system obtained by recursion (32) can easily be established either by substituting the above expressions for $q_i(\lambda)$ and $p_i(\lambda, \mu)$ in (50)–(51) or from the recursion relation (45). The result is

$$\begin{aligned} k_0^{(-3)} &= \frac{-(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2)}{\lambda_1^3 \lambda_2^3} & k_1^{(-3)} &= \frac{\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1^3 \lambda_2^3} \\ k_0^{(-2)} &= \frac{\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} & k_1^{(-2)} &= \frac{-(\lambda_1 + \lambda_2)}{\lambda_1^2 \lambda_2^2} \\ k_0^{(-1)} &= \frac{-(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} & k_1^{(-1)} &= \frac{-1}{\lambda_1 \lambda_2} \\ k_0^{(1)} &= \lambda_1 \lambda_2 & k_1^{(1)} &= -(\lambda_1 + \lambda_2) \\ k_0^{(2)} &= -(\lambda_1 + \lambda_2) \lambda_1 \lambda_2 & k_1^{(2)} &= \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 \\ k_0^{(3)} &= \lambda_1 \lambda_2 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) & k_1^{(3)} &= -(\lambda_1^3 + \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 + \lambda_2^3). \end{aligned} \quad (52)$$

The geodesic Hamiltonians γ_i have in our case the form

$$\gamma_0 = \frac{-\lambda_2}{\lambda_1 - \lambda_2} \frac{1}{2} \lambda_1 \mu_1^2 + \frac{-\lambda_1}{\lambda_2 - \lambda_1} \frac{1}{2} \lambda_2 \mu_2^2 \quad \gamma_1 = \frac{1}{\lambda_1 - \lambda_2} \frac{1}{2} \lambda_1 \mu_1^2 + \frac{1}{\lambda_2 - \lambda_1} \frac{1}{2} \lambda_2 \mu_2^2$$

so that they indeed have the form (44) with f_i of the form (12) (as has been pointed out at the end of section 2) and we can identify the functions $f_i(\lambda_i)$ in (12) as $f_i(\lambda_i) = \frac{1}{2} \lambda_i$ which is the form that can be used in order to solve the inverse Jacobi problem associated with equations (46) and in consequence to separate the system (29).

As a second example we will consider a quite generic (but still two-dimensional) bi-cofactor system with matrices G and \tilde{G} of the form

$$G = \begin{bmatrix} q_1^2 + 1 & q_1 q_2 \\ q_1 q_2 & q_2^2 \end{bmatrix} \quad \tilde{G} = \begin{bmatrix} 1 & q_1 \\ q_1 & 2q_2 \end{bmatrix}$$

It is no longer potential. The first few of the bi-cofactor systems produced by the recursion formulae (32) and (33) are defined by the quasi-potentials

$$\begin{aligned}
 k_0^{(-2)} &= \frac{4 + 2q_2 + 2q_2^2 + q_1^2}{q_2^2} & k_1^{(-2)} &= \frac{-(2q_1^2 - 2q_2^2 + q_1^2q_2 - 4q_2)}{q_2^3} \\
 k_0^{(-1)} &= \frac{2 + q_2}{q_2} & k_1^{(-1)} &= \frac{2q_2 - q_1^2}{q_2^2} \\
 k_0^{(1)} &= \frac{q_2^2}{2q_2 - q_1^2} & k_1^{(1)} &= \frac{q_2(2 + q_2)}{2q_2 - q_1^2} \\
 k_0^{(2)} &= \frac{q_2^3(2 + q_2)}{(2q_2 - q_1^2)^2} & k_1^{(2)} &= \frac{q_2^2(4 + 2q_2 + q_1^2 + q_2^2)}{(2q_2 - q_1^2)^2}
 \end{aligned} \tag{53}$$

and the corresponding forces $M^{(r)}(q)$ are

$$\begin{aligned}
 M^{(-2)} &= \frac{1}{q_2^4}(q_1(3 + q_2), q_2(4 + q_2))^T \\
 M^{(-1)} &= \frac{1}{q_1^3}(q_1, q_2)^T \\
 M^{(1)} &= \frac{-1}{(q_1^2 - 2q_2)^2}(q_1(1 + q_2), q_2^2) \\
 M^{(2)} &= \frac{q_2}{(q_1^2 - 2q_2)^3}(q_1(q_1^2 + 2q_2^2 + 4q_2 + 4), q_2(q_1^2 + 2q_2^2 + 2q_2))^T.
 \end{aligned}$$

Also in this case the solutions of (40) can easily be calculated:

$$\lambda_1(q) = \frac{(q_2 + 2 + \sqrt{\Delta})q_2}{2(q_1^2 - 2q_2)} \quad \lambda_2(q) = \frac{(q_2 + 2 - \sqrt{\Delta})q_2}{2(q_1^2 - 2q_2)}$$

with $\Delta = 4q_1^2 + (q_2 - 2)^2 \geq 0$. The coordinate curves given by these equations consist of the non-confocal ellipses and hyperbolas [26] and an arbitrary point (q_1, q_2) in the q -plane may lie not only on the intersection of an ellipse and a hyperbola (as was the case in the classical separability theory) but also on intersection of two ellipses or two hyperbolas. Formulae (42) for $\mu(q, p)$ are in this case too complicated to be presented. The inverse relationships are however still quite compact:

$$\begin{aligned}
 q_1 &= -2 \frac{\sqrt{-\lambda_1\lambda_2(\lambda_1 + 1)(\lambda_2 + 2)}}{\lambda_1 + \lambda_2 + \lambda_1\lambda_2} & q_2 &= -2 \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2 + \lambda_1\lambda_2} \\
 p_1 &= 2 \frac{\lambda_1\lambda_2(\mu_1\lambda_1 - \mu_2\lambda_2 + \mu_1 - \mu_2)}{\lambda_1 - \lambda_2} \\
 p_2 &= \frac{-(\lambda_1 + 1)(\lambda_2 + 2)(\mu_1(\lambda_1^2\lambda_2 - \lambda_1^2 + \lambda_1\lambda_2) - \mu_2(\lambda_1\lambda_2^2 - \lambda_2^2 + \lambda_1\lambda_2))}{(\lambda_1 - \lambda_2)\sqrt{-\lambda_1\lambda_2(\lambda_1 + 1)(\lambda_2 + 2)}}
 \end{aligned}$$

where we have chosen not to simplify the last expression since

$$-\lambda_1\lambda_2(\lambda_1 + 1)(\lambda_2 + 2) = q_1^2q_2^2/(q_1^2 - 2q_2)^2$$

is always non-negative. Applying these formulae we can express—after long algebraic manipulations—the geodesic Hamiltonians γ_i in the DN coordinates:

$$\begin{aligned}
 \gamma_0 &= \frac{-\lambda_2}{\lambda_1 - \lambda_2} 4\lambda_1^2(\lambda_1 + 1)\mu_1^2 + \frac{-\lambda_1}{\lambda_2 - \lambda_1} 4\lambda_2^2(\lambda_2 + 1)\mu_2^2 \\
 \gamma_1 &= \frac{1}{\lambda_1 - \lambda_2} 4\lambda_1^2(\lambda_1 + 1)\mu_1^2 + \frac{1}{\lambda_2 - \lambda_1} 4\lambda_2^2(\lambda_2 + 1)\mu_2^2
 \end{aligned}$$

so that $f_i(\lambda_i) = 4\lambda_i^2(\lambda_i + 1)$ in this case. The quasi-potentials (53) in the DN coordinates must attain the same form as the quasi-potentials (50)–(51) do, namely the form given by (52) since the change of variables (40)–(42) is designed so that the recursion (32) in the DN coordinates always attains the form (45).

Finally, let us consider a three-dimensional example with matrices G and \tilde{G} chosen as

$$G = qq^T + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} q_1^2 & q_1q_2 & q_1q_3 + 1 \\ q_1q_2 & q_2^2 + 1 & q_2q_3 \\ q_1q_3 + 1 & q_2q_3 & q_3^2 \end{bmatrix}$$

$$\tilde{G} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q^T + q \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & q_1 \\ 0 & 1 & q_2 \\ q_1 & q_2 & 2q_3 \end{bmatrix}.$$

The recursion formulae (32) and (33) applied to the geodesic flow $\dot{q} = 0$ yield an infinite sequence of the quasipotentials $k_\xi^{(r)} = \sum_{i=0}^2 k_i^{(r)} \xi^i$, $r = \dots, -1, 0, 1, \dots$. Some of them are

$$k_0^{(-1)} = \frac{2q_1q_3 + 2q_1 + 1}{2q_1q_3 + q_2^2 + 1} \quad k_1^{(-1)} = \frac{q_1(q_1 + 1)}{2q_1q_3 + q_2^2 + 1} \quad k_2^{(-1)} = \frac{q_1^2}{2q_1q_3 + q_2^2 + 1}$$

$$k_0^{(1)} = \frac{2q_1q_3 + q_2^2 + 1}{q_1^2} \quad k_1^{(1)} = \frac{2q_1q_3 + 2q_1 + 1}{q_1^2} \quad k_2^{(1)} = \frac{2 + q_1}{q_1}$$

$$k_0^{(2)} = \frac{(2 + q_1)(2q_1q_3 + q_2^2 + 1)}{q_1^3} \quad k_1^{(2)} = \frac{(2 + q_1)(4q_3 + 4 + 2q_1 - q_2^2)}{q_1^3}$$

$$k_2^{(2)} = \frac{q_1^2 - 2q_1q_3 + 2q_1 + 3}{q_1^2}.$$

The corresponding forces $M^{(r)}$ are

$$M^{(-1)} = \frac{-1}{(2q_1q_3 + q_2^2 + 1)^2} (q_1^2, q_2(q_1 + 1), q_1q_3 - 1)$$

$$M^{(1)} = \frac{-1}{q_1^3} (0, 0, 1)^T$$

$$M^{(2)} = \frac{-1}{q_1^4} (q_1^2, q_1q_2, q_1q_3 + q_1 + 3)^T$$

$$M^{(3)} = \frac{-1}{q_1^5} (q_1^2(q_1 + 4), q_1q_2(q_1 + 3), q_1^2q_3 + q_1^2 + 3q_1 + 6)^T$$

and they fast become complicated with the increasing $|r|$. In this case formulae (40)–(42) and their inverses are very complicated and can be handled only with the help of a computer algebra package. We will therefore quote here only the formulae for $q(\lambda)$,

$$q_1 = -2 \frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + 1}$$

$$q_2 = 2 \frac{\sqrt{-(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1 + \lambda_2 + \lambda_3 + 1)}}{\lambda_1 + \lambda_2 + \lambda_3 + 1}$$

$$q_3 = \frac{1}{4} \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1 + \lambda_2 + \lambda_3) - 3}{\lambda_1 + \lambda_2 + \lambda_3 + 1}$$

so that they are built of symmetric polynomials of order three in λ . The quasi-potentials $k_i^{(r)}$ presented above attain in the above DN coordinates the form

$$\begin{aligned}
 k_0^{(-1)} &= \frac{-(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)}{\lambda_1\lambda_2\lambda_3} & k_1^{(-1)} &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1\lambda_2\lambda_3} & k_2^{(-1)} &= \frac{-1}{\lambda_1\lambda_2\lambda_3} \\
 k_0^{(1)} &= -\lambda_1\lambda_2\lambda_3 & k_1^{(1)} &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 & k_2^{(1)} &= -(\lambda_1 + \lambda_2 + \lambda_3) \\
 k_0^{(2)} &= (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_1\lambda_2\lambda_3 \\
 k_1^{(2)} &= -(2\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2^2 + \lambda_1\lambda_3^2 + \lambda_2\lambda_1^2 + \lambda_2\lambda_3^2 + \lambda_3\lambda_1^2 + \lambda_3\lambda_2^2) \\
 k_2^{(2)} &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3
 \end{aligned}$$

which is in accordance with the recursion formula (45). The geodesic Hamiltonians γ_i have the following structure:

$$\begin{aligned}
 \gamma_0 &= \frac{\lambda_2\lambda_3}{\Delta_1}4(1 + \lambda_1)\mu_1^2 + \frac{\lambda_1\lambda_3}{\Delta_2}4(1 + \lambda_2)\mu_2^2 + \frac{\lambda_1\lambda_2}{\Delta_3}4(1 + \lambda_3)\mu_3^2 \\
 \gamma_1 &= \frac{-(\lambda_2 + \lambda_3)}{\Delta_1}4(1 + \lambda_1)\mu_1^2 + \frac{-(\lambda_1 + \lambda_3)}{\Delta_2}4(1 + \lambda_2)\mu_2^2 + \frac{-(\lambda_1 + \lambda_2)}{\Delta_3}4(1 + \lambda_3)\mu_3^2 \\
 \gamma_2 &= \frac{1}{\Delta_1}4(1 + \lambda_1)\mu_1^2 + \frac{1}{\Delta_2}4(1 + \lambda_2)\mu_2^2 + \frac{1}{\Delta_3}4(1 + \lambda_3)\mu_3^2
 \end{aligned}$$

so that they have exactly the form (44) with (12) and with $f_i(\lambda_i) = 4(1 + \lambda_i)$.

One can see that in all the above examples the functions $f_i(\lambda_i)$ do not depend on i , i.e. $f_i(\lambda_i) = f(\lambda_i)$.

6. Conclusions

In the present paper we performed separation of variables for the recently discovered class of quasi-potential systems called bi-cofactor systems. These systems generalize the classical potential systems with additional, quadratic in momenta, integral of motion in the sense that they reduce to these systems in the case when one of the matrices G and \tilde{G} is the identity matrix. In this special case the separation formulae (40)–(42) reduce to the well-known form of the classical separability theory formulae for separation of natural Hamiltonian systems by a point transformation. In the general case, however, these formulae do not have the form of a point transformation and are to our knowledge new in the literature. We concluded the study with some nontrivial examples in which the functions $f_i(\lambda_i)$ in (12) actually do not depend on i and we can make a conjecture that it is always the case in bi-cofactor systems.

Acknowledgments

We would like to thank S Rauch-Wojciechowski, H Lundmark and C Waksjö for interesting discussions during our work on this paper. We also thank the referees for useful remarks and references. MB was partially supported by KBN grant no 5P03B 004 20.

References

[1] Błaszak M 1998 On separability of bi-Hamiltonian chain with degenerated Poisson structures *J. Math. Phys.* **39** 3213
 [2] Błaszak M 1998 Bi-Hamiltonian separable chains on Riemannian manifolds *Phys. Lett. A* **243** 25
 [3] Błaszak M 1998 *Multi-Hamiltonian Theory of Dynamical Systems (Texts and Monographs in Physics)* (Berlin: Springer)

- [4] Błaszak M 1999 Theory of separability of multi-Hamiltonian chains *J. Math. Phys.* **40** 5725
- [5] Błaszak M 2000 Inverse bi-Hamiltonian separable chains *J. Theor. Math. Phys.* **122** 140
- [6] Błaszak M 2000 Separability of two-Casimir bi- and tri-Hamiltonian chains *Rep. Math. Phys.* **46** 35
- [7] Błaszak M 2000 Degenerate Poisson pencils on curves: new separability theory *J. Nonlinear. Math. Phys.* **7** 213
- [8] Falqui G, Magri F and Tondo G 2000 Reduction of bihamiltonian systems and separation of variables: an example from the Boussinesq hierarchy *Theor. Math. Phys.* **122** 176
- [9] Falqui G, Magri F and Pedroni M 2000 Bihamiltonian geometry and separation of variables for Toda lattices *Preprint nlin.SI/0002008*
- [10] Błaszak M 2002 From bi-Hamiltonian geometry to separation of variables: stationary Harry–Dym and the KdV dressing chain *J. Nonlinear Math. Phys.* **9** 1–13
- [11] Marciniak K and Rauch-Wojciechowski S 1998 Two families of nonstandard Poisson structures for Newton equations *J. Math. Phys.* **39** 5292–306
- [12] Rauch-Wojciechowski S, Marciniak K and Lundmark H 1999 Quasi-Lagrangian systems of Newton equations *J. Math. Phys.* **40** 6366–98
- [13] Lundmark H 1999 Integrable nonconservative Newton systems with quadratic integrals of motion *Linköping Studies in Science and Technology Theses No 756* Linköping University
- [14] Lundmark H 2001 A new class of integrable Newton systems *J. Nonlinear Math. Phys.* **8** 195–99 supplement—*Proc. NEEDS'99 (Kolymbari, Crete)*
- [15] Marciniak K and Rauch-Wojciechowski S 2001 Integrable perturbations of the harmonic oscillator and Poisson pencils *Inverse Probl.* **17** 191–209
- [16] Eisenhart L P 1934 Separable systems of Stäckel *Ann. Math.* **35** 284–305
- [17] Benenti S 1997 Intrinsic characterization of the variable separation in the Hamilton–Jacobi equation *J. Math. Phys.* **38** 6578–602
- [18] Ibort A, Magri F and Marmo G 2000 Bihamiltonian structures and Stäckel separability *J. Geom. Phys.* **33** 210–28
- [19] Morosi C and Tondo J 1997 Quasi-bihamiltonian systems and separability *J. Math. Phys.* **38** 6578–602
- [20] Kalnins E G and Miller W Jr 1980 Killing tensors and variable separation for Hamilton–Jacobi and Helmholtz equations *SIAM J. Math. Anal.* **11** 1011–26
- [21] Crampin M and Sarlet W 2001 A class of non-conservative Lagrangian systems on Riemannian manifolds *J. Math. Phys.* **42** 4313–26
- [22] Magri F 1978 A simple model of the integrable Hamiltonian equation *J. Math. Phys.* **19** 1156–62
- [23] Lundmark H 2001 Newton systems of cofactor type in Euclidean and Riemannian spaces, *Linköping Studies in Science and Technology Dissertations No 719* Linköping University
- [24] Falqui G, Magri F, Pedroni M and Zubelli J P 2000 A bi-Hamiltonian theory for stationary KdV flows and their separability *Regular Chaotic Dynamic* **5** 33–52
- [25] Ramani A, Dorizzi B and Grammaticos B 1982 Painlevé conjecture revisited *Phys. Rev. Lett* **49** 1539–41
- [26] Lundmark H 2001 Private communication